5. Vectors

Objectives

By building on experiences, to learn the nature of vectors in 1, 2 and 3 dimensional spaces including

- their mathematical descriptions,
- their various representation,
- addition and subtraction of vectors, both graphically and analytically,
- vector multiplications (scalar and vector) - mathematical description and physical interpretation.

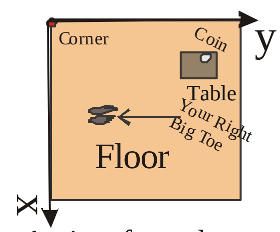
You will be able to

- describe and define the features of vectors,
- represent vectors graphically and mathematically in 1, 2 and 3 Dimensions,
- add and subtract vectors both graphically and analytically,
- do vector multiplications and understand their physical interpretation.

THE EVENT

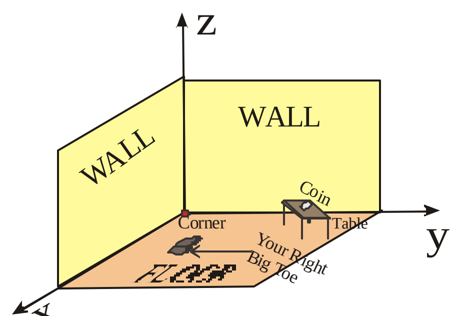
You are in a room. Put a table with coin on it on an arbitrary location on the floor.

1. Using the walls of the room, describe how to get from the tip of your right big toe to the coin.
2. Describe the location of the tip of your right big toe relative to one corner in the room.
3. Describe the location of the coin relative to the same corner.
4. Describe the location of the coin relative to your toe.
5. Describe any relationship(s), if any, between 2, 3 and 4.
6. Make a sketches of the situations.



A view from the top

Figure 1



ROOM

Figure 2 The Event 3D view

CONTINUALLY REVISIT THIS EXERCISE AS YOU GO THROUGH THIS CHAPTER ON VECTORS

5.1. Some Basic Ideas on Vectors

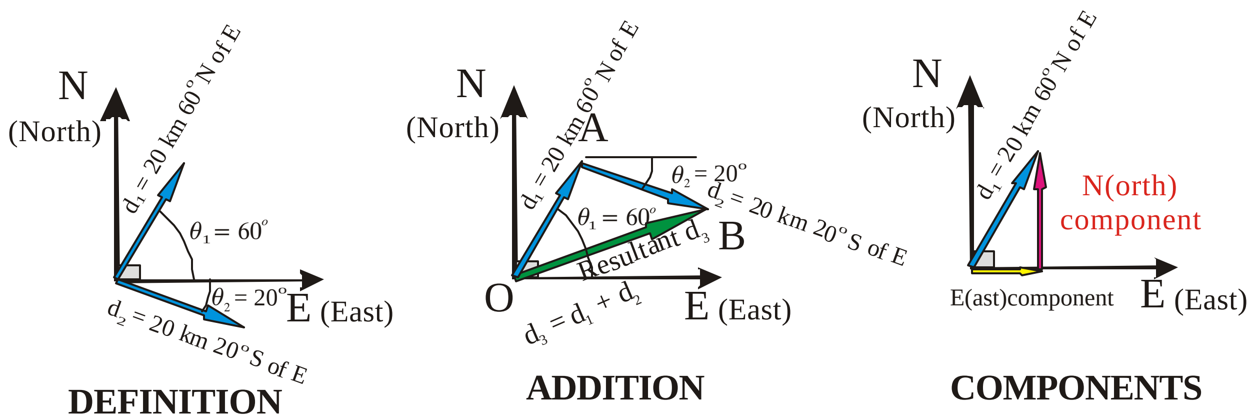


Figure 3

Using the trip, Fig. 3 above, shows the main features of vectors , their addition and their components. In fact, the trip can be considered the quintessential or prototypical vector. Let's now examine these pictures.

Representation: As shown in "DEFINITION", a vector is pictorially represented by an arrow. The arrow head or tip is the end, the tail the beginning. The arrow points in the direction of the physical action. In the trip model of the vector, the tail is the beginning of the trip, and the tip, the end of the trip.

Addition: The middle picture, "ADDITION", shows how to add vectors. A total trip often consists of a series of trips, each a different distance in a different direction. In such a series of trips, the end of one trip is the start of the next trip. The total or net trip is the sum of the series of individual trips. It is the difference between where you ended and where you started. Viewing vectors as trips, and the representation of a vector as an arrow with its tail the start of the trip and the head (point) as the end, shows that vector addition is a tip-to-tail, tip to tail operations - adding the pieces by putting the tip of the first vector at the tail of the second. So **vectors add "tip to tail"** *-join the tail of this vector to the tip of the last vector,* ***or*** *join the tip of this vector to the tail of the next.*

Reference Systems: In order to deal with vectors quantitatively, we first must set up a reference system. A reference system consists of

1. an origin, a starting point,
2. and a set of coordinate axes centered on the origin with one axis for each spatial dimension.

Starting at the origin, if traveling in one direction is called the positive direction, the opposite direction is called the negative direction. All of the above figures show the reference system - mutually perpendicular coordinate axes with North the positive direction for one, and East the positive direction for the other.

Components: They are a set of vectors,

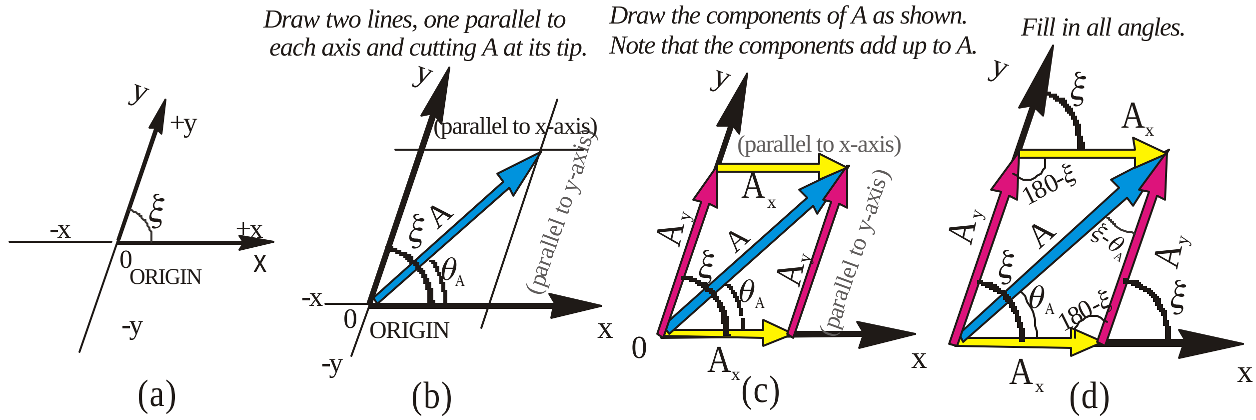
1. one parallel to **each** of the coordinate axis,
2. whose vector sum equals the original vector.

The third picture, "Components", shows the North and East components of d_1.

5.2. 2-Dimensional Vectors.

5.2.1. Coordinate Axes can be at Any Angle , but 90º is BEST

Let's start with 2-Dimensional vectors. How do we set up coordinate axes for 2-Dimensional vectors? The best way to answer this question is as follows. Knowing that we need two axes, the question is "at what angle should axes have relative to each other?".



RELATIONSHIP OF THE COMPONENTS

1. Law of Cosines $A^2 = A_x^2 + A_y^2 + 2A_x A_y \cos\xi$ (note the angles in the lower triangle in (d)

2. Law of Sines $\frac{A}{\sin(180-\xi)} = \frac{A_x}{\sin(\xi - \theta_A)} = \frac{A_y}{\sin\theta_A}$. So, $A_x = \frac{A\sin(\xi-\theta_A)}{\sin(180-\xi)} = \frac{A(\sin\xi\cos\theta_A - \cos\xi\sin\theta_A)}{\sin\xi}$ (a), and $A_y = \frac{A\sin\theta_A}{\sin(180-\xi)} = \frac{A\sin\theta_A}{\sin\xi}$ (b).

Figure 4 *General 2D coordinate system - axes at any angle to each other*

Figure 4 develops the case for 2-D vectors when the coordinate axes make an arbitrary angle, ξ, relative to each other. Since $\sin 90^o = 1$ and $\cos 90^o = 0$, these relationships become very simple when the angle $\xi = 90^o$. The component triangles become right triangles. The Law of Cosines, Eq 1. , reduces to the Pythagorean Theorem ;

$$A^2 = A_x^2 + A_y^2 + 2A_x A_y \cos 90^o = A_x^2 + A_y^2 + 2A_x A_y * 0 = A_x^2 + A_y^2$$

And the Law of Sines, Eq.2(a). gives

$$A_x = \frac{A\sin(x - q_A)}{\sin(180\text{-}x)} = \frac{A(\sin x \cos q_A - \cos x \sin q_A)}{\sin x} = \frac{A(\sin 90^o \cos q_A - \cos 90^o \sin q_A)}{\sin 90^0}$$

$$= \frac{A(1*\cos q_A - 0*\sin q_A)}{1} = A\cos q_A \ .$$

And 2(b) gives

$$A_y = \frac{A\sin q_A}{\sin x} = \frac{A\sin q_A}{\sin 90^o} = \frac{A\sin q_A}{1} = A\sin q_A.$$

So, when the coordinate axes are mutually perpendicular (at 90^o to each other) the components become independent of each other. Then

$A_x = A\cos\theta_A$, $A_y = A\sin\theta_A$ and $A^2 = A_x{}^2 + A_y{}^2$ (Pythagorean Theorem).

A we will see later, this independence of the components also applies in 3-D when the coordinate axes are mutually perpendicular to each other. Such coordinate systems are called “Normal” or “Orthogonal” systems. This component independence and the Pythagorean Theorem are true in all such systems regardless of how many dimensions they have.

Exercise : Get the Components for 2-D vector shown in an oblique coordinate system
Graphically get the x and y components of the vector A relative to the oblique coordinate system as shown in (a).

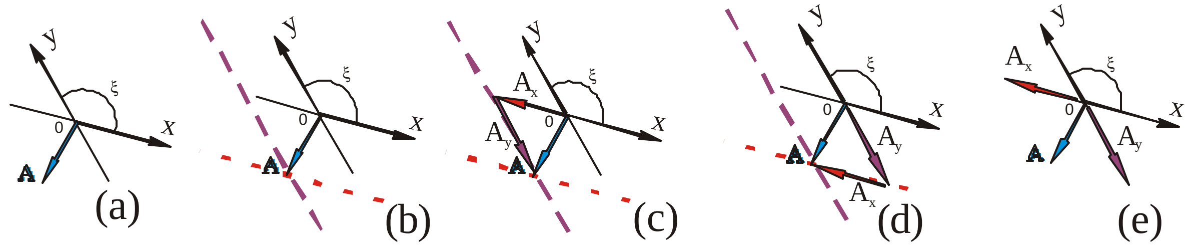


Figure 5

Getting the components of a 2-D vector in an oblique coordinate system is actually quite easy. The steps are shown in the above figure. (a) shows the vector and its oblique coordinate system.

The first step, as shown in (b), is to sketch two lines, one parallel to each coordinate axis and touching the tip of the vector A.

The second step is to draw the components A_x and A_y as shown in (c). Or, you can make the set shown in (d). Both satisfy the definition of components.

They can then be arranged as shown in (e).

5.2.2 Components for 2-D with mutually perpendicular coordinate axes

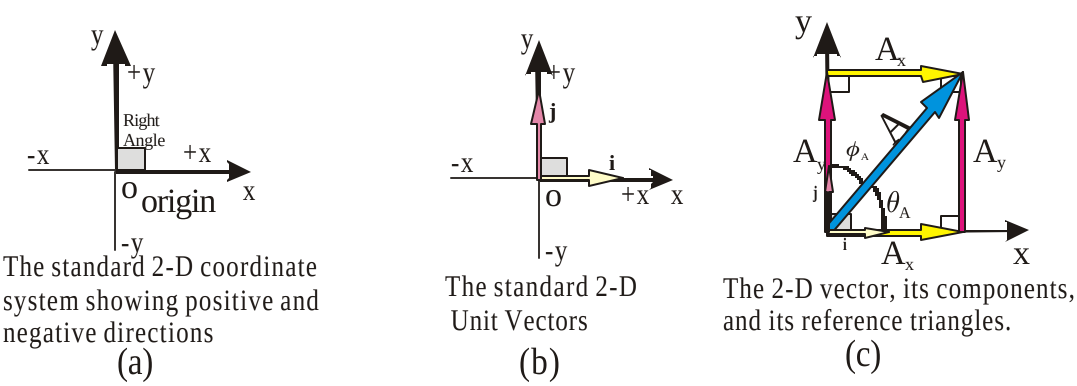


Figure 6

Fig. 6 develops the 2-D orthogonal (normal) coordinate system, unit vectors and reference triangles for getting the components of a vector.

Unit vectors are a set of vectors, one parallel to each coordinate axis, and each having a magnitude equal to one unit. The standard is that the unit vector **i** is for the x-axis, and **j** for the y-axis, as shown in Fig 6 (b)..

The **reference triangle**s are the two triangles contained in the rectangle formed by the components of the vector, as shown in Fig. 6 (c).

REVISIT THE EVENT - use any new ideas that you have learned.

5.2.3 Values of the Components

There are two basic methods to find the components of 2-D vectors. One method uses the reference triangle. The other is is more general, it uses the full circle. The angle of the vector is relative to the +x axis.

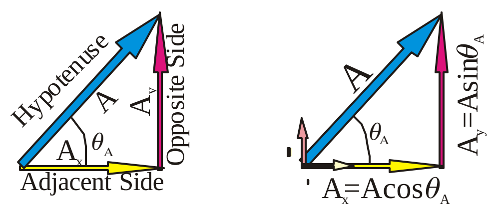
5.2.3.1 Values of the Components using the Reference Triangles - Two Views

The reference triangles provide two views of the components. They are reversed views, specifically adjacent and opposite sides switch positions. In Fig. 7 where the angle θ_A is relative the x-axis, A_y is the opposite to the angle, and A_x is adjacent. But, in Fig. 8 where the angle ϕ_A is relative the y-axis, A_x is the opposite to the angle, and A_y is adjacent - just the reverse of Fig. 7. The sign of the components are handles manually, that is by you..

Angle Relative to the x-axis

$$\sin\theta_A = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{A_y}{A}$$

$$\cos\theta_A = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A_x}{A}$$



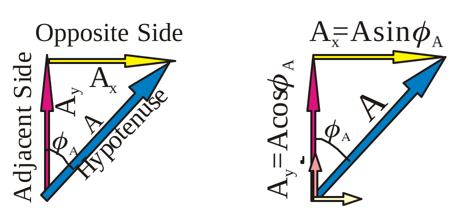
THE DEFINING TRIANGLE VALUES OF THE COMPONENTS

Figure 7

Angle Relative to the y-axis

$$\cos\phi_A = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A_y}{A}$$

$$\sin\phi_A = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{A_x}{A}$$



THE DEFINING TRIANGLE VALUES OF THE COMPONENTS

Figure 8

WARNING As a result of this reversal situation in the reference triangle method, do not blindly memorize $A_x = A\cos\theta_A$ and $A_y = A\sin\theta_A$, first check your reference triangle.

5.2.3.2 Values of the Components - General Full Circle

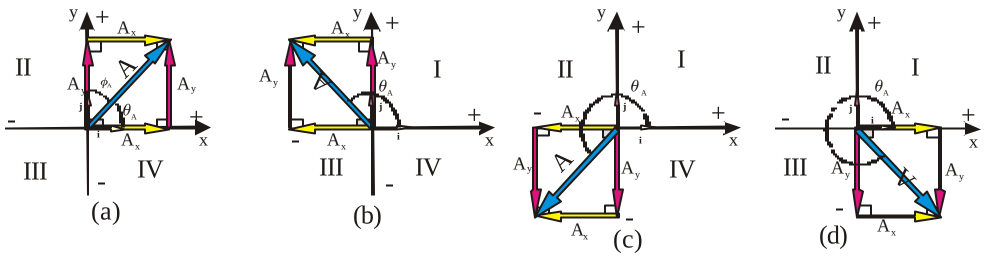


Figure 9

In this general form the angle θ_A, is relative to the +x axis. Here $A_x = A\cos\theta_A$ and $A_y = A\sin\theta_A$ that is obvious in the (a) for the first (I) quadrant extends to all quadrants. The sign of the component is automatically taken into account by the sign of the cos and sin functions. So, A_x is + , positive, when A in in quadrants I and IV, and -, negative, when A lies in quadrant II and III. While A_y is +, positive, when A lies in quadrants I and II, and -, negative, when A lies in quadrants III and IV.

Exercise 1: Fill in the table.

Quadrant of A	sign of A_x	sign of A_y

5.2.3.3 Representations:

Vectors can be written, more formally put, represented, in several different forms. Four are shown below.

Free Form $A_x =$, A_y = ; **Unit vectors** $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j}$

Ordered Pairs $\mathbf{A} = (A_x, A_y)$ **Polar** **A** = A **g** θ_A relative to +x-axis .

In the Polar representation A is the magnitude of the vector and θ_A is the angle relative to the + , positive x- axis as shown in Fig. 8.

A NOTE ON NOMENCLATURE

Every vector has a name, for example A, D, F_1, F_k , T_3 . Entities associated with a vector must carry its name. For instance, if θ is the angle of a vector, then you must specify which vector , θ_A ,θ_D , θ_{F1} , θ_{Fk} , or θ_{T3} . If you have a series of T's, like, T_1, T_2 , T_3 , then it is permissible a short cut, θ_1 , θ_2 , and θ_3 .

Exercise 2. For each of the vectors below the angles are for the reference method. Do the following.

a. Find the components of each of the vectors both graphically. Use a ruler with mm marking and a protractor. Draw an orthogonal coordinate system on an 8.5"x11" sheet of paper. Draw the vector to scale on the coordinates. Draw the components as in Fig.5c. Measure the components.

b. Calculate the components using the general method of Fig. 8, and both reference triangles in Fig.6 and Fig.7.

c. Write each vector in the four above representations.

1. A = 10 cm in quadrant II, $\phi_A = 30^o$. 2. A = 10cm in quadrant I, $\theta_A = 30^o$

3. A = 10 cm in quadrant III, $\phi_A = 30^o$. 3. A = 10cm in quadrant IV, $\theta_A = 30^o$.

(Partial Answers 1. 10/_120^o , (-5.00,8.66)cm; 2. (8.66,5.00)cm; 3. (-8.66,-5.00)cm; 4. (8.66,-5.00)cm.

REVISIT THE EVENT - use any new ideas that you have learned.

5.2.4 ADDITION and SUBTRACTION of VECTORS

5.2.4.1 ADDITION of VECTORS

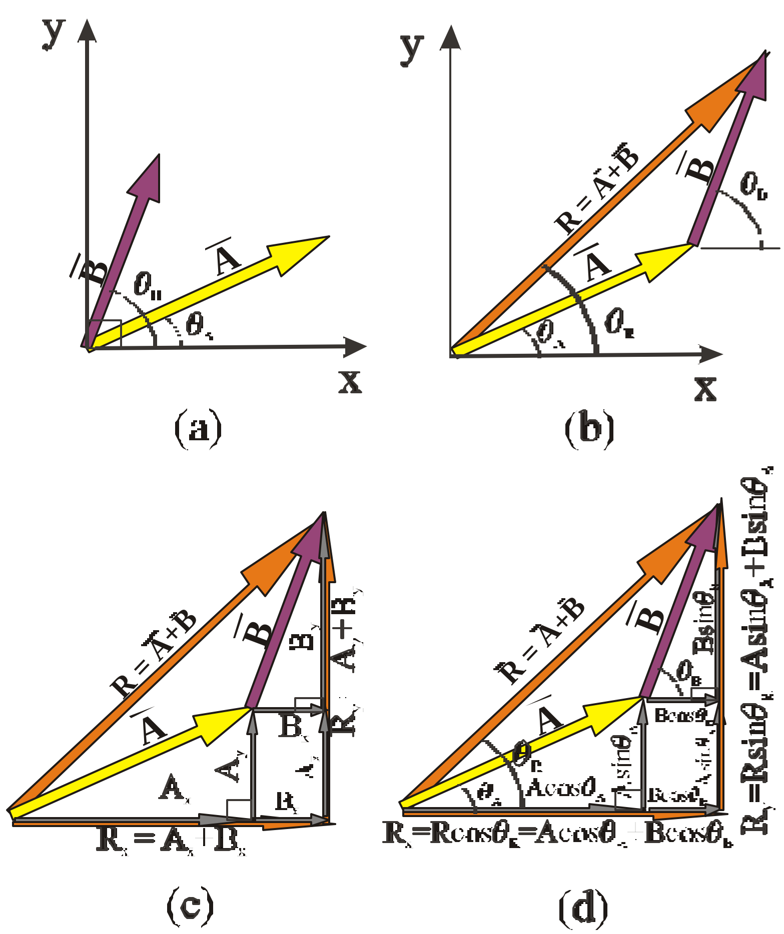


Figure 10

Vectors are added tip-tail as shown in Fig. 3. Figure 10, below, shows the evolution of the addition of two vectors.

(a) shows the two vectors A and B.

In (b) the vector B is moved so its tail is at the tip of A for the tip-to-tail setup. The resultant R=A+B is drawn.

In (c) the components of each of the three vectors are drawn and named, i.e., A_x, etc

Finally, in (d) the values of each component is written in terms of the magnitudes and angle of the parent vector.

Then finally for magnitude of R, the Pythagorean Theorem gives

$$A = \sqrt{A_x^2 + A_y^2}\ .$$

The angle θ_R results from the fact that $\tan\theta_R$ = Opposite/Adjacent = R_x/R_y , hence

$$\theta_R = \arctan\,(R_y/R_x)\ .$$

Exercise 3. Replicate the steps in Fig. 9 in detail to generate $R_2 = B + A$. Review Figures 3, 5c, and 6. What relationships do you discover between A+B and B+A? Explain.

5.2.4.2 SUBTRACTION of VECTORS

Subtracting vectors is much like subtracting numbers, but with very different results. Figure 11 shows two methods of subtracting vectors.

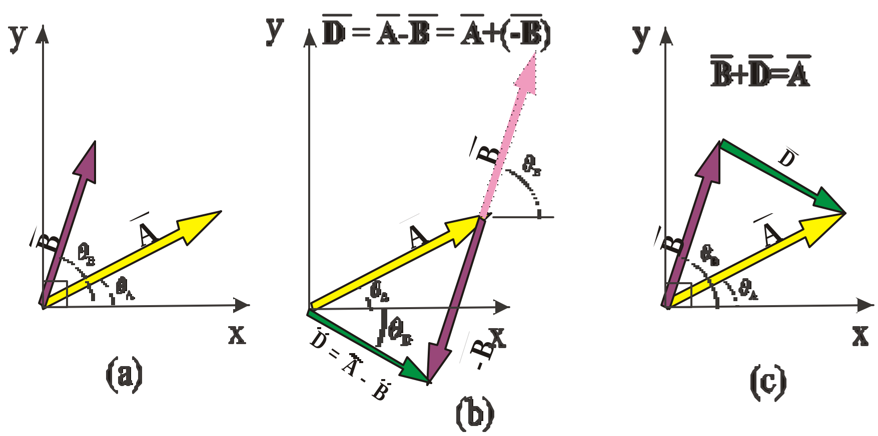


Figure 11

One, (b), is by using the fact that A - B = A + (-B). First, reverse B to get -B, then add (-B) to A. This is similar to the situation in numbers where subtracting a given number from another number is the same as adding the negative of that given number to the other number.

The second, (c), evolves from asking the question, "to get D=A-B, "what vector D added to B gives A?." Hence, D is added to B to get A; B+D(?) = A.

Exercise 4. Replicating the steps in Fig. 11, do D_2 = B-A . What do you discover about any relationships between A-B and B-A? Explain.

REVISIT THE EVENT - use any new ideas that you have learned.

5.3 VECTORS IN 3-DIMENSIONS

5.3.1 Components

3-D vectors are more complicated. Let's start by looking at the standard orthogonal 3-D coordinate system and its unit vectors in Fig. 12.

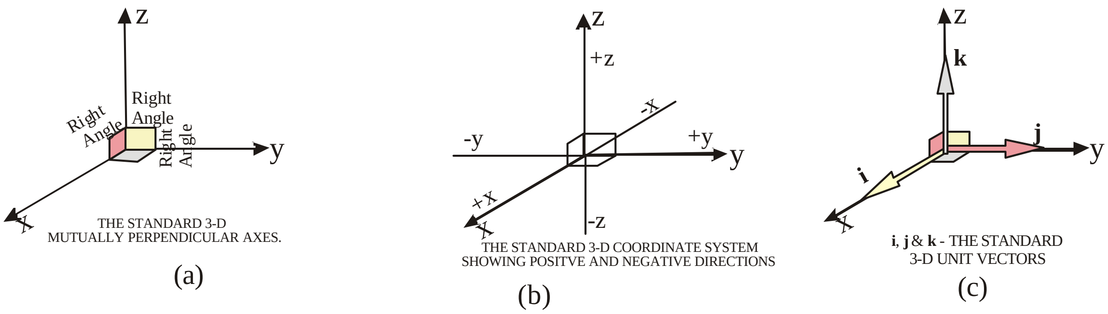


Figure 12

This is similar to Fig. 5(a) and 5(b), but much more complicated because the 3-D is represented in a 2-D plane. So, before discussing a 3-D vector, let's put the coordinate system in a more familiar environment, like setting up the 3-D coordinate of Figure 11c in the corner of a room, or on a box. This is shown in Fig.13.

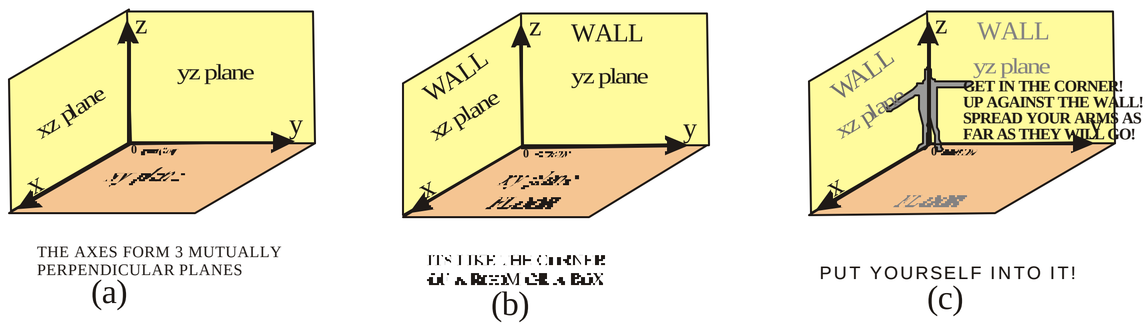


Figure 13

Now let's embed a vector into the 3-D orthogonal coordinate system. The Fig. 14, below, shows a vector A in 3-D and develops the resolution of A into its A_x , A_y , and A_z components.

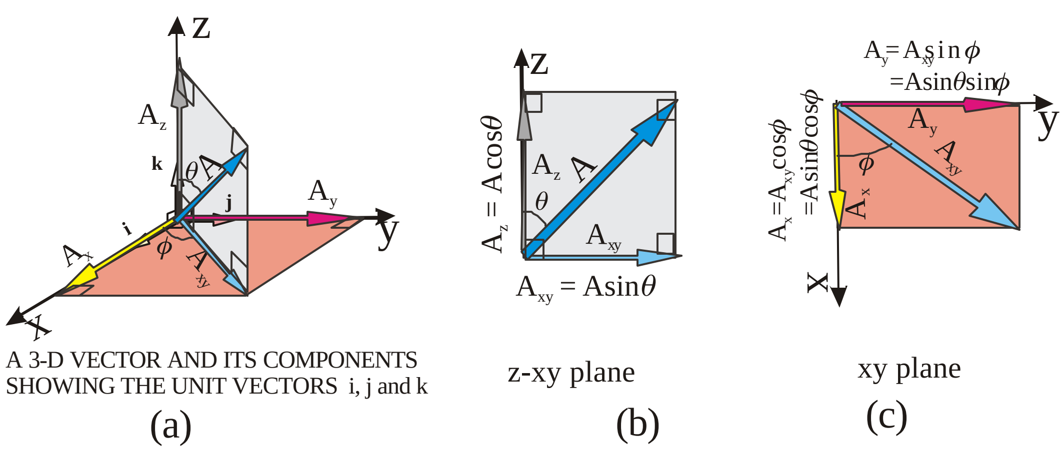


Figure 14

The vector A and its component decomposition are in 14(a). Fig. 14(b) shows the plane defined by A_{xy} and the z-axis. It is needed to calculate the projection of **A** onto the xy plane. There we see that $A_{xy} = A\sin\theta$ and $A_z = A\cos\theta$. Fig.14(c) shows the view when looking down parallel to the z-axis(careful, don't let the z-axis poke you in the eye). It contains the xy plane with the projection of **A** onto that plane, A_{xy}, and the components A_x and A_y. There we find that , $A_x = A\sin\theta \cos\phi$, and $A_y = A\sin\theta \sin\phi$

, respectively. Finally Fig. 15 shows how the components add to give the original vector A.

This path, **A** = **A**$_x$ + **A**$_y$ + **A**$_z$, is not the only one. Any order of adding the three components gives the same result.

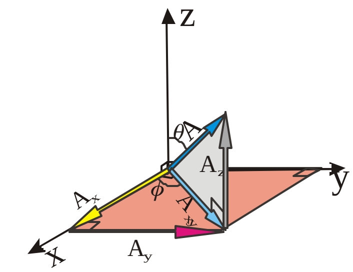


Figure 15

Exercise

a. Show that there are six (6) possible ways to add the three components to get **A**.

b. Using the corner of a box, do the other five (5) possible orders of adding the three components. You can use materials like pieces of spaghetti for **A** and its components. You can use paste, putty or flour paste to hold **A** in its position.

5.3.2 REPRESENTATIONS

Free Form	A_x = A_y = A_z =	Ordered Pairs	**A** = (A_x ,A_y, A_z)
Unit vectors	**A** = A_x **i** + A_y **j** + A_z **k**	Polar	A = A /_θ,ϕ

5.3.2.1 Example:

1. Find the components of A = 0.10 /_30°,60° m .

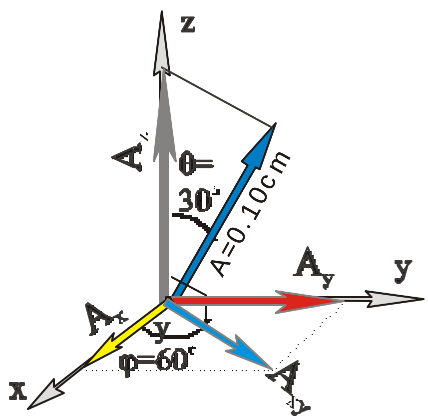


Figure 16

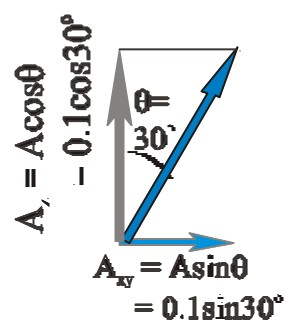


Figure 17

First find A_z , the component parallel to the z-axis, and A_{xy} , the component in the xy plane. See Fig.17.

A_z = Acosθ = 0.10cos30° = 0.10*0.866 = 0.0866 m.

A_{xy} = Asinθ = 0.10sin30° = 0.10*0.500 = 0.0500 m.

Now, by Fig.18, get A_x , the component parallel to the x-axis, and A_y , the component parallel to the y axis by using A_{xy} in the xy plane.

A_x = A_{xy}cosϕ = Asinθcosϕ =0.10*sin30°cos60° = 0.10* 0.050*0.500 =0.0 25 m.

A_y = A_{xy}sinϕ = Asinθsinϕ = 0.10*sin30°sin60° 0.10*050*0.866 =0.0 433 m.

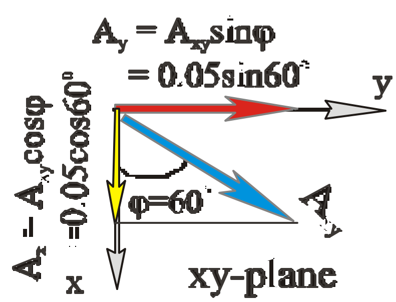


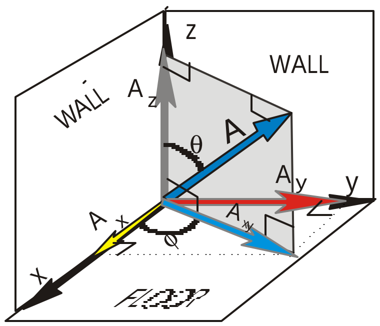
Figure 18

So A = (A_x,A_y,A_z) = (0.025,0.0433,0.086) m
= 0.025 **i** + 0.0433 **j** + 0.0866 **k** m
= 0.10 /_30°,60° m . .

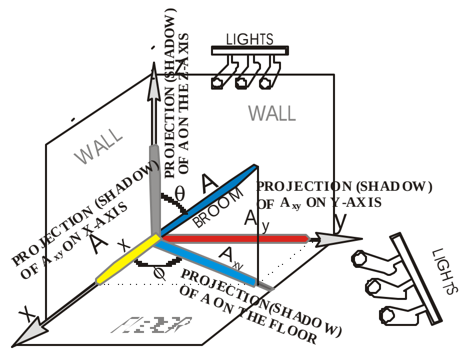
REVISIT THE EVENT - use any new ideas that you have learned.

5.3.2.2 Setting Up the Components in the Corner of a Room or Box

Let's look at the room corner more carefully. Now let's put some light on the situation and look at shadows.



PUT THE VECTOR IN THE CORNER OF A ROOM OR A BOX



PUT A BROOM IN THE CORNER AS THE VECTOR. THEN SHINE LIGHTS FROM ABOVE, FROM THE SIDES AND LOOK AT THE SHADOWS.

Do this shadow play with a broom in the corner of a room as shown here. It might be easier to set it up in the corner of a small box, like a shoe box or smaller. Use spaghetti, straws, whatever, for the axes and vector **A**. You can use things like clay, flour paste, putty or plaster to hold **A** in place. Then use a flash light in the different positions shown in the picture.

5.3.3 MULTIPLICATION OF VECTOR

There are two types of multiplication of vectors - the **scalar product** and the **vector product**. The scalar product is a multiplication between two vectors that yields a scalar. The vector product is a multiplication between two vectors that yields a third vector.

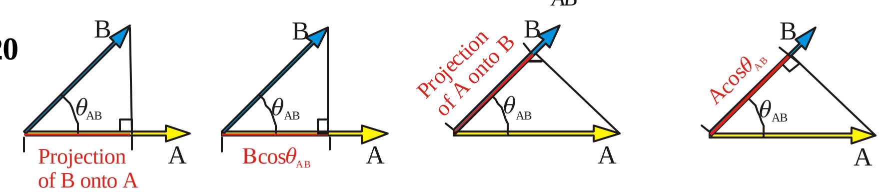
5.3.3.1 The Scalar Product.

This product is useful in "projection" type situations. Projections are shadows as shown in the 3-D vector pictures on the previous pages. They occur in many applications such as fluid flow , heat flow, illumination of surfaces by light beams, shadow effects, electricity and magnetism. Two examples are shown below.

Scalar or Dot Product $\mathbf{A} \bullet \mathbf{B}$

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\theta_{AB}$$

Figure 20



Examples:

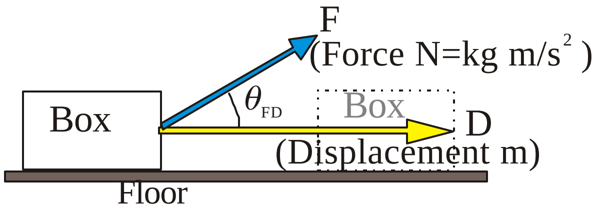


Figure 21

WORK

A box is pulled by an external agent applying a force F while moving the box over a displacement D as shown. The work done by the external agent on the box is

$$\text{Work} = \mathbf{F} \bullet \mathbf{D} = FD\cos\theta_{FD}$$

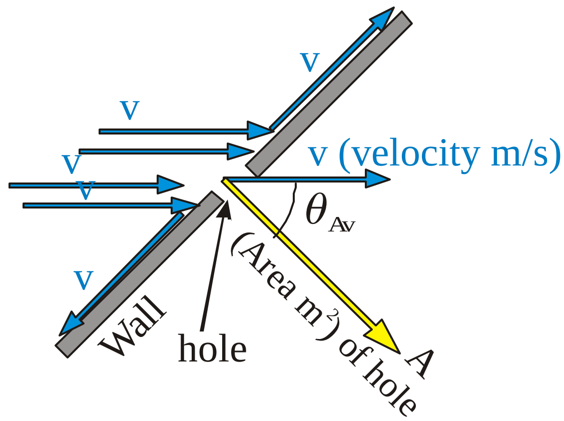


Figure 22

FLUID FLOW

A fluid flows with a velocity v towards a wall that has a hole of area A and makes an angle θ_{Av} to the direction of v. The quantity Q (m^3/s) of fluid that flows through the hole is

$$Q = \mathbf{A} \bullet \mathbf{v} = Av\cos\theta_{Av} \ (m^3/s)$$

The fluid that hits the wall changes direction to run parallel to the wall, as shown.

Unit Vectors

Since the unit vectors are parallel to themselves then

$$i \bullet i = j \bullet j = k \bullet k = \cos 0^o = 1$$

$$i \bullet j = i \bullet k = j \bullet k = \cos 90^o = 0$$

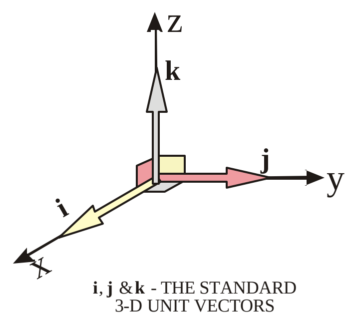


Figure 23

Example

Given

A = 3i - 5j   and   B = -i + 3j

Find   **A • B**

$$A \bullet B = (3i - 5j) \bullet (-i + 3j)$$
$$= 3i \bullet (-i) + 3i \bullet (3j) + (-5j) \bullet (-i) + (-5j) \bullet (3j)$$
$$= 3*(-1) i \bullet i + 3*3 i \bullet j + (-5)*(-1) j \bullet i + (-5)*3 j \bullet j$$
$$= -3 i \bullet i + 9 i \bullet j + 5 j \bullet i + -15 j \bullet j$$

= -3 + 0 + 0 + (-15)

= -3 -15 = -18.

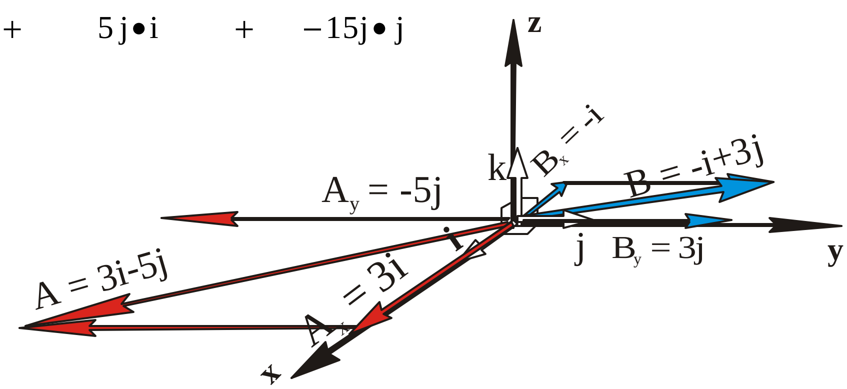
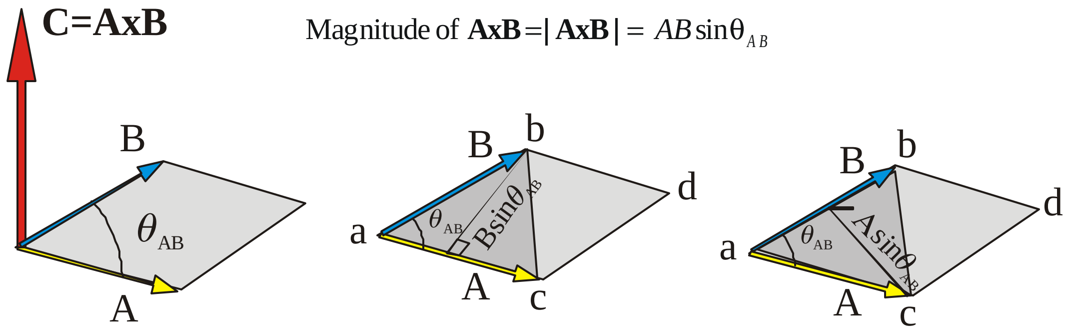


Figure 24

5.3.3.2 The Vector Product

The Vector or Cross Product **AxB**

Magnitude of **AxB** = |**AxB**| = $AB\sin\theta_{AB}$



C ⊥ **A**; **C** is perpendicular to **A**.
C ⊥ **B**; **C** is perpendicular to **B**.
So, **C** is perpendicular to the plane containing **A** and **B.**

Figure 25

Area of triangle $abc = AB\sin\theta_{AB} = \frac{1}{2}|\mathbf{AxB}|$.
Area parallelogram abcd =2 x Area of parallelogram *abcd*.
So,area parallelogram *abcd* =| **AxB** |.

This product is extremely important in all rotational and torque situations. This is because torque and rotation are relative to an axis perpendicular to the plane of the force and lever arm in torque, and the plane of the motion in rotation. It is also important in electricity and magnetism.

You get the direction of the vector C by the **Right hand Rule**. "Stick the fingers of your right hand in the direction of the first vector, **A** , curl them into the direction of the second vector, **B** . Stick out your thumb. It points in the direction of **C** = **AxB** .

Below are examples of the Vector Product. These examples all relate to relationship between linear quantities, force F, linear displacement Δs, linear velocity v, linear acceleration a and angular quantities, torque τ, angular displacement Δθ, angular velocity ω, and angular acceleration α . Note that in each situation the linear quantity F, Δs, v and a, are at the tip of the position vector r_O . To carry out the Vector Product, slide this vector back to the origin point O so that the two vectors are joined at their tales - r_O and F for τ_O , r_O and Δs for Δθ, and so on.

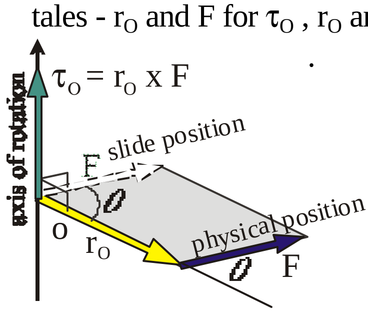
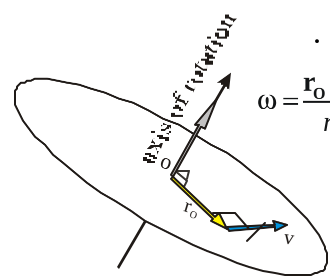


Figure 26

Torque τ_O about the perpendicular axis through the point O due to a force F a position r_O from the axis through O and at an angle θ with r_O



$\omega = \dfrac{\mathbf{r_O} \times \mathbf{v}}{r_O^2}$

Figure 27

A point at position r_O from the axis of rotation, spinning about that axis with angular velocity ω, has linear velocity v at r_O as shown

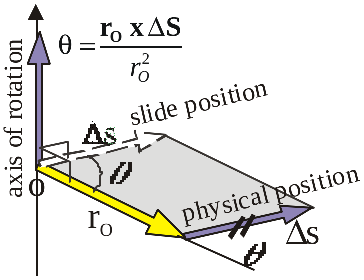


Figure 28

Angular displacement θ due to a linear displacement Δs at a position r_O from the axis of rotation.

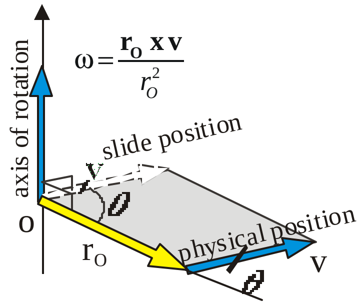


Figure 29

Angular velocity ω due to a linear velocity v at a position r_O from the axis of rotation.

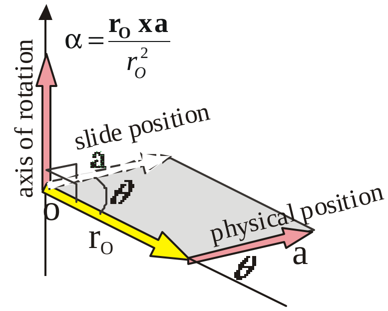


Figure 30

Angular acceleration α due to a linear acceleration a at a position r_O from the axis of rotation.

In the equations in Figures 26 to 29, the vector product in the numerator gives the direction of the resulting vector. The r_o^2 in the denominator makes the units and magnitudes correct. Check them out.

Unit Vectors

Since the unit vector are parallel to themselves, then

$$i \times i = j \times j = k \times k = \sin 0^o = 0$$

The unit vectors are mutually perpndicular, then using the right hand rule we get

and $$i \times j = k \qquad j \times k = i \qquad k \times i = j$$

$$j \times i = -k \quad k \times j = -i \quad i \times k = -j$$

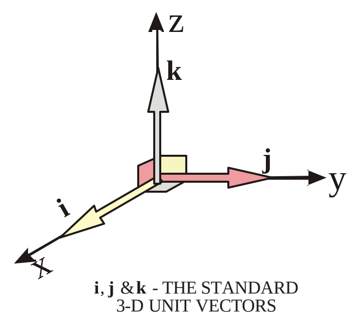


Figure 31

Example

Given

A = 3i - 5j and B = -i + 3j

Find

AxB = (3i - 5j) x(-i +3j)
= 3ix(-i) + 3ix3j -5jx(-i) + (-5j)x3j
= -3(ixi) +9(ixj) + 5(jxi) - 15(jxj)
= -3*0 + 9 k - 5 k - 15*0
= 0 +4k - 0
= 4 k.

REVISIT THE EVENT - use any new ideas that you have learned.

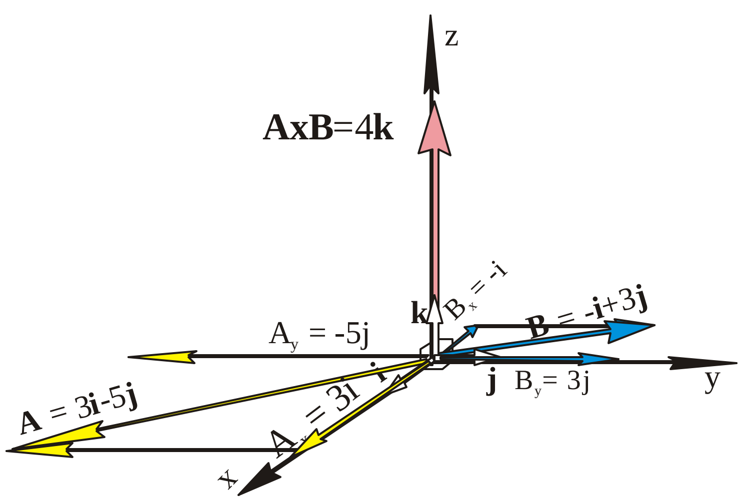


Figure 32